Sam: Cooper
Donum Authoris

A VINDICATION OF THE *Miscellanea Analytica*: In ANSWER to a late PAMPHLET ENTITLED OBSERVATIONS, &c.

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A

VINDICATION, &c.

THE Author of this Reply, thinks he cannot begin better, than by laying before his Readers the Occaſion of it, and explaining to them his Intention in the following Obſervations.

The ingenious Author of the *Miſcellanea Analytica*, having declar'd himſelf a Candidate for the Mathematical Profeſſorſhip now vacant in this Univerſity; preſented to the Electors, that part of his Performance, which has given occaſion to the Remarks at preſent under conſideration. The Severity with which this Specimen of his Abilities has been treated, in a late Treatiſe, entitled, *Obſervations*, &c. and the little, or rather no reaſon, which appear'd on a further examination of this Work, to have tempted the Author of it, to make his private Miſtakes and Miſapprehenſions, a public topic, induc'd a *friendly* hand to take part in the Debate; and to endeavour, by the following Vindication, to make known the real Uſefulneſs of the Work itſelf; and the Falſity, or Futility ,of all the Objections that have been brought againſt it.

The plauſible appearance which the Author of theſe *Obſervations* puts on, in the

firſt pages of his book, flatter'd us with the hopes of the ſame Candour, and Impartiality, in the further proſecution of his Enquiry; and we were content that he ſhould, as the nature of his ſubject requir'd, commend and cenſure, with freedom and indifference: and tho' we were a little alarm'd with his firſt inſinuation of the neglect the Book might have fallen into, had it appear'd at any other time; yet we were again reliev'd, when with a ſeeming tenderneſs and concern he told us, that "if "the Occaſions of Cenſure were more nu- "merous than thoſe of Commendation, we "were not to conſider them as more agree- "able to him."

If this was his diſpoſition, we are at a loſs to account for thoſe illiberal Reflections, and ſevere Expreſſions, which occur ſo frequently in his *Obſervations*: for had his Sentiments been dictated by Truth, or had his Strictures proceeded from a love of Science only; it would have been ſufficient to have ſhewn the *young Analyſt* his Miſtakes, with that temper at leaſt, which his public endeavours in the Cauſe of Learning entitled him to expect. But while there appears ſuch Malevolence in the execution, we need not enquire into the deſign of his Remarks. Be that to their Author: we ſhall find ſufficient employ in detecting his Errors; and in doing this we ſhall conduct ourſelves

ourſelves with that decency, which if ever a propriety of character is to be obſerv'd, ought eſpecially to obtain among the Sons of Learning: and tho' our *Obſervator* ſeems frequently to have forgot that humanity; which if not the ſtudy of Philoſophy, at leaſt, a nobler ſtudy ought to have taught its *Profeſſors*; yet we ſhall purſue our Enquiry without inſult, or reflection; and think ourſelves happier in refuting his Arguments with Candour, than in retorting his Irony with Succeſs.

There ſeems to be but two general Methods of reducing Equations (above cubics) to thoſe of fewer Dimenſions; one of which is, to aſſume ſome two or more Equations at Pleaſure, (whoſe Roots are ſuppos'd to be the ſame with thoſe in the given Equation; and which multiplied together, form an Equation of the ſame Dimenſions with that propoſed) and then by equating the homologous Terms, to find the Coefficients of thoſe Equations of fewer Dimenſions: But this is *Des Cartes*'s method, or at leaſt ſimilar to it; and therefore the Lemma is true in this caſe. The other is by aſſuming ſome lower Equation, whoſe Roots are ſuppoſed to be the ſame with thoſe in the given one, and (dividing the given Equation by it) to put the Remainder and Quotient, each equal to nothing, but this is in Fact the ſame thing: for ſuppoſe the E-

quation

quation to be reduced, be a Biquadratic, and we divide it by a ſimple Equation, whoſe Root is ſuppoſed to be one of the Roots of the Biquadratic: It may be any of them, and conſequently, the Remainder in which this Root, and given Quantities are only concerned, muſt be a Biquadratic, agreeable to the Lemma. The reaſoning is the ſame, let the dividing Equation be what it will; and therefore the Lemma is true in this Method alſo.

It may poſſibly be objected here, that the Method of Solution uſed by our Author in his 5[th] Cap. is different from either of thoſe here laid down, and yet general; but, I believe, it will appear, to the intelligent Reader, to amount much to the ſame Thing; for that Method of reſolving Biquadratics is juſt the ſame, as multiplying theſe two Quadratics ($x^2 + \overline{a - b}.\, x + c - d = 0$; and $x^2 + \overline{a + b}.\, x + \overline{c + d} = 0$) together, and comparing the Equation thence ariſing with the general Biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$, as any one will find upon Trial.

We are told that Biquadratics may be reduced to Quadratics by a cubic Equation, each of whoſe Roots have two Values, with their Signs only changed; that is (if I rightly underſtand the Expreſſion) an Equation of the third Power, that has ſix Roots. This, according to all our Algebra-

iſts,

iſts, ancient, as well as modern, is impoſſible; for they tell us that the Number of Roots in any Equation, is always equal to the Number of Units, in the higheſt Power of that Equation. Dr. *Saunderſon* indeed ſays, that an Equation of this kind ($x^6 - ax^3 + b = 0$) is called a Quadratic, becauſe by ſubſtituting y, for x^3, it may be reduced to one; but after we have found the Roots of the Equation, formed by ſuch a Subſtitution, we ſhall have the Roots of a cubic Equation to extract, in order to come at thoſe of the Equation firſt propoſed; and therefore he can only mean, that, as this Equation is in Form of a Quadratic, we may reduce it to a Cubic by compleating the Square.

This very obvious Method of reducing Equations, by means of the ſimple ones of which they are compoſed, goes upon this ſuppoſition, that we can find thoſe ſimple ones. So we can in particular Caſes. But our Author expreſſly tells us (pag. 41,) that the Lemma does not hold unleſs the Method of Solution be general. He likewiſe there ſhews us, how there happens to be that ſeeming Contradiction, between what is aſſerted in the firſt Lemma, and his Reduction of Biquadratics, namely, becauſe the Quadratic has four Roots, *i. e.* it is in fact a Biquadratic.

There is indeed a miſtake in the Solution

tion of the ſecond Lemma, which however, is far from being ſo conſiderable, as it is here repreſented to be. For it will eaſily appear to any one, who knows the compoſition of Equations, that the Index n of the higheſt Power in the Equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3}$, &c. $=0$. can never at all be concerned in the Series for finding the Sum of the n^{th} Power of the Roots; for let the Equation be what it will, the ſecond Coefficient will always be the Sum of the Roots; the third, the Rectangle under every two, &c. and it is from theſe Coefficients, (without any Reſpect to the Dimenſions of the Equation) that this Series is formed. This Error therefore cannot make the Series there given, particular, as is aſſerted by the Author of theſe *Obſervations*.

The Laws which the Series obſerves, as laid down by our Author, are beſt explained by an Example. Suppoſe then, that $\overline{n-6}$ is the Index of the power of the Coefficient of one Dimenſion p, then the literal Products annexed to this, will be v, qs, q^3 and r^2; becauſe v is the Coefficient of 6 Dimenſions, q of 2, and s of 4; conſequently their Rectangle makes a Coefficient of 6 Dimenſions, and ſo of the reſt. The Coefficients prefixed to thoſe Rectangles, according to our Author's Rule, will be n, $n.\overline{n-5}$, $n.\frac{\overline{n-5}}{2}.\frac{\overline{n-4}}{3}$ and $n.\frac{\overline{n-5}}{2}$, all which exactly agree

gree with those Terms of the Series given, whose Index of the Power of p, is $\overline{n - 6}$: But I fancy our *Observer*, by one Term in a Series, understands all the different Products that are multiplied into the same Power of the Coefficient p, when he says, "that if the Series be formed according to "so much only of the Law as is here laid "down, it will be wrong in every step "after the third". Though this be frequently meant by one Term in a Series, yet it appears from the Law of the literal Products laid down by our Author, that it could not be his meaning in this Place. For unless he meant there to take in all the different Products that could make up a power, equal to that which is subtracted from n in the exponent of the Power of p, it had been necessary for him to tell, which particular one was to be taken in. If then this be his meaning as to the Law of the first, it is evident from his assuming one particular Product (*viz.*) $p^{n-m}. q^{\alpha}. r^{\beta}. s^{\gamma}. t^{\delta}.$ &c. to shew us how to find the Coefficient, that every different Product must have a different Coefficient. I call this $p^{n-m}. q^{\alpha}. r^{\beta}. s^{\gamma}.$ &c. a particular Product, because though, those Characters α, β, γ, δ &c. may represent any affirmative integer Quantities whatever, they can but each of them represent one at the same Time; and yet p^{n-m} may have several different Powers and Combinations of

the Quantities *q*, *r*, *s* &c. joined with it, as appears from the above.

The Author of the *Obſervations* objects to the firſt Corollary to this Propoſition; becauſe (as he ſays) it is uſeleſs: for any Man who is taught to find thoſe Quantities, may add them together without having a Series given him for that Purpoſe; this objection may with equal propriety be employed in ſhewing the little Uſe, any general Methods of obtaining the ſums of Serieſes are of; for any common man would think, that when he had the Law by which a Series was formed, (from which he could find the Terms themſelves) that he could add them together without having a general expreſſion found for their ſum; and yet we ſee many (and thoſe too of the greateſt Skill in theſe Sciences) have employed themſelves chiefly to theſe kind of Enquiries.

The ſecond Corollary, it is true, will be of no Uſe in finding the ſum of a Series whoſe firſt Term is the Sum of any Number of Quantities, the ſecond the Sum of their Squares, the third the ſum of their Cubes &c. in infinitum. (for which it ſeems chiefly intended) unleſs all thoſe Quantities be leſs than 1, and greater than —1; yet the *Obſerver* was a little too haſty when he aſſerted, that this reaſoning would lead us into any Abſurdity, ſuppoſing the Roots of the Equation were without theſe Limits; for

for, if the Numerator of any Fraction be conſtant, the Fraction increaſes as the Denominator decreaſes; and when the Denominator becomes o, the Quantity becomes infinite: whence we may conclude (by the very ſame Method of Reaſoning that is uſed by the great Sir *Iſaac Newton* in his Principia) that when the Denominator becomes Negative, the Quantity becomes more than infinite. Or thus: A Quantity (as Dr. *Saunderſon* tells us) may from infinite affirmative flow down to finite affirmative; and ſo by paſſing through o, become finite Negative: therefore the reciprocal of this Quantity will be firſt o, then finite affirmative, then infinite; and at laſt more than infinite. It may here be ſaid, that from infinite it becomes finite negative; but let it be conſidered, that as the firſt Quantity is continually upon the increaſe, its reciprocal muſt always decreaſe by the ſame ſteps; and conſequently, unleſs we ſuppoſe, (contrary to every Body) that a negative Quantity is greater than o, we muſt be obliged to allow, that its reciprocal muſt be greater than the reciprocal of o, which is infinite.

Tho' the Aſſertion that $\frac{0}{0}$ is equal to 4, may appear a grand Abſurdity to a Perſon not much acquainted with theſe mathematical Myſteries; yet, as the Author of the Remarks has been too cautious to affirm it,

I can hardly perſuade myſelf but he knew, that Dr. *Saunderſon* in the 469th Article of his Algebra, has expreſsly told us, that $\frac{0}{0}$ may be equal to any finite Quantity whatever; which muſt therefore be determined from the particular Nature of the Problem. To make this intelligible to any Perſon who underſtands the Reduction of a ſimple Equation. That o multiplied into any finite Quantity whatever, gives o for the Product, is an Axiom denied by Nobody: Let then *a* repreſent any Quantity, and it will be $a \times 0 = 0$; and conſequently $a = \frac{0}{0}$. The moſt common Method of determining this Quantity is, by taking the Fluxions of the Numerator and Denominator, and dividing them one by the other, as in the Caſe before us,) where the Value of $\frac{p - \overline{p^{n+1}}}{1 - p}$ is to be determined, when $p = 1$ and $n = 4$; and conſequently, both the Numerator and Denominator vaniſh at the ſame time:) By the Method above laid down we ſhall have $\frac{\dot{p} - \overline{n+1} \cdot p^n \dot{p}}{-\dot{p}} = \overline{n+1} \cdot p - p$ equal (becauſe $p = 1$) $n = 4$. It *may* here be objected that I firſt make p a variable Quantity, and then I ſuppoſe it equal to a determinate one. I find the ſame thing done by *Simpſon*, *Hayes*, *Landen* &c. upon the very ſame occaſion; in whoſe Works, thoſe who want information,

tion, may find the Reaſons for this Operation.

The Author of the *Obſervations* does not indeed in the leaſt ſeem to have underſtood the Meaning of the 3d. Lemma; and therefore, we cannot allow him to judge of its Uſe. For, according to his Explanation, it is no more, than to take all the Combinations of 2, 3, 4, *&c.* of the given Powers of the Roots, and multiplying them together, to find the Product. This is very different from the Intention of our Author, as expreſſed in the Lemma, where he propoſes to find the m Power of one Root multiply'd into the p Power of every other, *&c.* But as it has been ſo greatly miſunderſtood by the Author of the Remarks, it may not be improper to give a ſhort Explanation of it. It is thus, To find the Sum of the m^{th} Power of one Root, into the p^{th} Power of *another*, into the q^{th} Power of *another*, *&c.* all the different Ways that they can be taken; or to find the Aggregate of all the different Combinations that can be made of the m^{th} Power of one Root, into the p^{th} Power of *another*, *&c.* Here it may be obſerv'd by the Way, that the firſt Exception made in the Remarks, is evidently imply'd in the Lemma itſelf.

For

For explaining the Method by which our Author finds the Quantities *A*, *B*, *C*, *&c.* we ſhall, (for the ſake of Perſpicuity) take an Example. Let it be the ſecond of thoſe given by our Author to this Lemma, where it is propoſed to find the Sum of each Root, multiply'd into the Square of another, into the Cube of another, all the different Ways that they can be put together. Then *A* is equal to the continual Product under the Sum of the Roots; the Sum of their Squares, and the Sum of their Cubes; *B* is equal to the Aggregate of the Sum of the Cubes of the Roots, ſquared, the Rectangle under the Sum of the 4th Powers, and the Sum of the Squares, and the Rectangle under the Sum of the 5th Powers, and the Sum; becauſe all the different Combinations of every two of the Powers propoſed; are, $\overline{1+2}$, $\overline{1+3}$, and $\overline{2+3}$, that is, 3, 4 and 5, and conſequently, as the Sum of each of thoſe Powers of the Roots, is to be multiplied (according to the Rule given us by our Author) into the $6-\overline{1+2}$, $6-\overline{1+3}$, and $6-\overline{2+3}$, Powers of the Roots reſpectively, the Value of *B* will be as above laid down, the Value of *C* is the Sum of the 6th $(1+2+3)$ Powers of the Roots.

The ſecond Exception taken notice of in the *Obſervations*, is made by the Author himſelf, in theſe Words immediately following

lowing the Examples, " Cum duo vel tres " vel plures Poteſtates ſint æquales, dividatur Summa per Producta 1.2.3.4. &c. " ut in præcedente Lemmate." Though it be poſitively aſſerted, that not a Hint is given of any ſuch Exception. The Reaſon of this Aſſertion I leave the Reader to determine.

It is ſaid, we ſhould have been told, how to change the Solution when n is an odd Number. It does not appear to us to want any changing. The Solution is general; and therefore will hold when applied to any particular Equation, whether the higheſt Power of that Equation be an odd or an even Number. Suppoſe, for Example, it be required to determine an Equation, whoſe Roots are each equal to the Sum of two of the Roots in this Cubic $x^3+3x^2-16x+12=0$, whoſe Roots are 1, 2, and —6. Then by forming an Equation from the general one given by our Author, we ſhall find it to be in this Caſe $v^3+6v^2-7v+60=0$, whoſe Roots are 3, —5 and —6, i. e. $\overline{1+2}$, $\overline{1-6}$, and $\overline{2-6}$; but the *Obſervator*, I ſuppoſe means, that if we tranſform the Equation by ſubſtituting (as we are taught) $-a-v$ for x, the Equation found, by ſuch a Transformation will not anſwer the Conditions of the Problem; becauſe when n is an odd Number, the higheſt Power of the unknown Quantity v will

will be Negative. It will ſo: for the Equation will be juſt the ſame as that above found, with all the Signs changed, which, it is evident, does not at all affect the Value of the Roots.

If the deſign of the *Obſervations* was to point out, to the Author of the *Miſcellanea Analytica*, the Faults in the 1ſt Chapter of his Book, as the Writer of them ſeems to inſinuate, it had been neceſſary for him to ſhew, in what particular Caſes the Solution of the 4th Problem fails; for to find out this, appears to us a much more difficult Taſk, than to diſcover the Author's Meaning in that Lemma, which is propoſed (in the 22d Page of the *Remarks*) as an inſuperable Difficulty.

What is done by *Des Cartes* and his Commentators, though it might probably ſuggeſt the Problems here ſolved, could never be of the leaſt Aſſiſtance in the Solution of them; for there is no more connection between Problems which teach us how to increaſe or diminiſh the Roots by given Quantities, and thoſe which ſhew us how to find an Equation, each of whoſe Roots are the Sum of the $n-1$ Roots of a given Equation, or to find an Equation each of whoſe Roots are the Squares of the Roots of a given Equation, than there is between the Reſolution of a Quadratic and that of a Biquadratic Equation; and yet we preſume that there are ſeveral Perſons who can

can with eaſe reduce a Quadratic Equation, that would be much puzzled to find the Roots of a Biquadratic.

Thoſe Maſters, indeed, could by their Methods deſtroy any *one* Term in an Equation, except the laſt, by an Equation of fewer Dimenſions than the given one. This ingenious Author can (whatever may be aſſerted to the cõntrary) by his Method deſtroy any *two* Terms in an Equation: of which he has given us an Example in the 5th Cap. where he takes away the 2d and 4th Terms, and by that means brings a Biquadratic into the Form of a Quadratic.

The general Reſolution of a Problem, is always the Rule to which we are to apply all particular Caſes; and therefore a Rule given in Words at Length ſeems unneceſſary.

It is no Wonder that our Author ſhould call Problems, that is, Propoſitions, in which ſomething is required to be found, Lemmas or Corollaries, according as they are neceſſary to the demonſtrating ſomething that follows, or are immediately deducible from the Propoſition going before, ſince ſo many Hundreds (and thoſe too of the greateſt Accuracy) have done the ſame Thing before him, without its being once objected to.

Though it was inconſiſtent with the Conciſeneſs with which the Lemma in the 21ſt Page of our Author, is expreſs'd, for it to be clear enough to be underſtood by a Perſon juſt entering upon the firſt Rudiments of Algebra; yet to any Body but moderately skill'd in theſe Studies, the Deſign of the Lemma, and the Reaſoning it is founded upon, will appear evident, not only from the Example which follows it, but even from the Nature of the Propoſition itſelf. We ſhall however endeavour to make it a little more intelligible, for the Benefit of thoſe who find ſo much Difficulty in comprehending it. Our Author ſays, that Algebraic Queſtions in general, are either Rules which expreſs the Relation between certain Quantities, or they are Theorems which ſhew us, by what Means we are to know how many Quantities of a given Kind, (*i. e.* a propoſed Kind) are contained in a given Equation. The Method of demonſtrating thoſe of the firſt Kind, immediately preſents itſelf to the Mind of any thinking Man, *viz.* to try whether there be any ſuch Relation or not; and this is the Method of demonſtrating, *tentando:* which does not therefore mean to demonſtrate a Propoſition by *repeated* Trials. To give an Example, *Mac Laurin* (in § 118, p^t. 1. of his Algebra,) tells us that, if $a^m - b^m$ be multiplied by $a^{n-m} + a^{n-2m} b^m + a^{n-3m} b^{2m}$, &c. continued till the Number of Terms be

be equal to $\frac{n}{m}$, the Product will be $a^n - b^n$. He demonſtrates this Propoſition by actually multiplying thoſe Quantities, *i. e.* by trying, whether there be the Relation propoſed between them or not.

By this Method may the third Lemma be demonſtrated; ſo that, though it is objected to our Author, that he has not given us a Demonſtration of this Propoſition; yet it cannot (at leaſt with Juſtice) be ſaid, that he has not ſhewn us how to come at one.

It is evident, that by Quantities of the given Kind, in the ſecond Kind of Rules to be demonſtrated, is meant Quantities of any propoſed Kind; and conſequently, by Quantities not of the given Kind, the contrary of theſe. For the Author tells us, that the Equation is to be drawn into a Quantity of the given Kind, and into another of the Kind not given; and then, if the firſt Multiplication increaſes the Number of Marks, by which we are to find the Number of Quantities of the given Kind in the Equation by Unity; and the latter neither increaſes no diminiſhes them, the Rule is true; and will hold in all Caſes: the Reaſon of this is evident; for, as the Number of Quantities of the given Kind are increaſed by Unity, the Marks by which we are to determine that Number, muſt be increaſed by Unity too; in the ſame Manner the latter muſt neither increaſe nor diminiſh them: again, if the latter Multiplication neither

increaſes nor diminiſhes them; and the former either has the ſame Number or increaſes them by Unity, the Rule is true; but not accurate, *i. e.* perfect: for as the Quantity of a given Kind muſt be generally expreſs'd, if we can, by ſubſtituting different Values of the ſame Kind; either make it to increaſe the Number of Marks, by which the Number of Quantities of the given Kind are to be found, or not to increaſe them, it is evident, that the Rule may ſometimes find the Number, and ſometimes not; and therefore, if it be propoſed thus, that all the Quantities which ſuch a Method does find are of the given kind, the Rule may be true, though from the Reaſoning above it appears, that it will not, in all Caſes, find *all* the Quantities of the given Kind contained in the Equation: This is evidently the Meaning of the Lemma, which is not only conſonant to Reaſon, but what muſt neceſſarily be underſtood by every Perſon (in the leaſt acquainted with theſe Matters) who reads it.

As to the general Inſinuations which run through the whole Compoſition, it is impoſſible to refute them till they are applied to particular Inſtances, which we are perſuaded would have been done, had it been poſſible.

We

We are unwilling wholly to paſs by the Commendation given by our *Obſervator* in his 17th Page, though we are forced to obſerve, that *there* his great Candour has evidently obliged him to betray his Judgement.

And here we ſhould naturally end our Remarks; but hope that the Reader will indulge us in a few *Obſervations* on the Concluſion of this Performance, where the Author has reſumed the ſame Appearance of Benevolence, with which he began it. He allows the Writer of that Book, he had been ſo induſtriouſly expoſing, Ability to correct Miſtakes, cauſed by Haſte and Inattention; but one cannot help ſmiling at his Inſinuation of the Laboriouſneſs of the Taſk. Thoſe Readers who are able and willing to examine the Merits of theſe Performances, muſt be alſo ſenſible of the many Errors, which muſt unavoidably attend a Publication of this ſort; and will be more generous, than to impute to the Author, Miſtakes which may be more reaſonably ſuppoſed to have proceeded from the Tranſcriber and the Preſs.

As to the Advice which he has ſo confidently given in his laſt Page, we hope the Contents of this will have made it needleſs for us to ſay much on that Head; for if our Readers are but convinced of the Juſtneſs *of our Obſervations*, they will rather

wiſh

wiſh to haſten than ſuſpend the Publication of the Book in Queſtion; — and the more ſo, as the Author of it has found a Friend to vindicate Him from the groſſer Part of the Imputation; and conſequently need not ſpend much Time in acquiring a needleſs Attention, to correct ſuch trivial Errors, as the Candour of the moſt learned Reader, will readily excuſe.

THE END.